ARTICLES

Photon position operator with commuting components

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A photon position operator with commuting components is constructed, and it is proved that it equals the Pryce operator plus a term that compensates for the adiabatic phase. Its eigenkets are transverse and longitudinal vectors, and thus states can be selected that have definite polarization or helicity. For angular momentum and boost operators defined in the usual way, all of the commutation relations of the Poincaré group are satisfied. This new position operator is unitarily equivalent to the Newton-Wigner-Pryce position operator for massive particles. [S1050-2947(99)02902-9]

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I. INTRODUCTION

In the quantum mechanics, each physical observable O is represented by a Hermitian operator $\hat{O}$. The possible results of a measurement of O are the eigenvalues of $\hat{O}$. If a particle's position is to be measured, positions in space $\mathbf{r}'$ should be the eigenvalues of some position operator, $\hat{\mathbf{r}}$. The eigenvectors $|\mathbf{r}'\rangle$ of $\hat{\mathbf{r}}$ define the position space wave function, $\psi(\mathbf{r}')=\langle\mathbf{r}'|\psi\rangle$. In a strict sense, the very existence of such a wave function requires a Hermitian position operator with commuting components. For the photon, no such position operator has previously been found.

The search for a photon position operator has a long history. In 1933, Pauli stated that the nonexistence of a density for the photon corresponds to the fact that the position of a photon cannot be associated with any operator in the usual sense [1]. Pryce [2] examined a number of definitions of the mass center in the context of field theories where an energy-momentum tensor exists. For the photon, he found the position operator

$$\hat{\mathbf{r}}_P=i\hbar\nabla_{\mathbf{p}}-i\hbar\frac{\mathbf{p}}{2p^2}+\hbar\frac{\mathbf{p}\times\mathbf{S}}{p^2}, \tag{1}$$

which does not have commuting components. Here $\nabla_{\mathbf{p}}$ is the gradient operator in $\mathbf{p}$ space and the components of the spin operator $\mathbf{S}$ are the three-dimensional generators of the $SO(3)$ group.

Newton and Wigner sought localized states that become, after a translation, orthogonal to the undisplaced localized states and for which operators of the Lorentz group apply [3]. While they arrived at a general expression for a position operator for massive particles and for zero mass particles of spin 0 or $\frac{1}{2}$, for zero mass particles with higher but finite spin, they found no such operator.

Wightman [4] developed the theory of localizability in a region based on imprimitive representations of the Euclidean group, and again came to the conclusion that photons are not localizable. Jauch and Piron [5] and Amrein [6] relaxed the requirement to generalized imprimitivities and concluded that particles, which may exist in superpositions of states of different helicity (photons but not neutrinos), can be localized in arbitrarily small volumes.

It is currently thought that photons are only weakly localizable, although single-photon states with arbitrarily fast asymptotic falloff of energy density do exist [7]. A lack of strict localizability is directly related to the absence of a position operator and a position space photon wave function, as previously discussed.

In the present paper, a Hermitian position operator with commuting components is systematically constructed. The resulting operator is then compared to the Pryce operator in Sec. II and the phase-invariant derivative in Sec. III. In Sec. IV, angular momentum and boost operators are defined, and it is proved that they provide a realization of the Poincaré group. Finally, the relationship of this position operator to others in the literature and to the existence of a photon wave function is discussed. *Système International* units are used throughout.

II. PHOTON POSITION OPERATOR

The eigenvectors will be assumed for simplicity to be three-vectors as in [8] and [9]. This can be extended to six component vectors [2,10] and includes vectors of the form $\sqrt{\epsilon}(\mathbf{E}\pm ic\mathbf{B})$ [10,9] for electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Alternatively, the vector potential $\mathbf{A}$ can be used and a generalization to four-vectors made [11]. The scalar product is assumed to be of the form

$$\langle\psi|\phi\rangle=\sum_i\int d^3p\,\psi_i^*\phi_i/(pc)^{2\alpha}, \tag{2}$$

so that a normalized free particle wave function goes as p^{α}. Normalization is covariant if $\alpha=\frac{1}{2}$, the usual choice in field theory [12].

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The procedure used here is to first construct a position operator with transverse eigenvectors, and then investigate its properties. This is done in **p** space where the natural first guess for a position operator is $i\hbar\nabla_{\mathbf{p}}$. If the eigenvectors are transverse to the direction of propagation, then the difficulty is that $\nabla_{\mathbf{p}}$ changes the direction of these vectors, mixing in a longitudinal component. One can simply subtract these terms, canceling the unwanted effects, and leaving only the eigenvector multiplied by the position of the photon. The effect of differentiating p^a must also be subtracted.

The two transverse unit vectors will be called $\mathbf{e}_{\mathbf{p}1}$ and $\mathbf{e}_{\mathbf{p}2}$, while $\mathbf{e}_{\mathbf{p}3}\equiv\mathbf{p}/p$ is the longitudinal unit vector. Thus the **p**-space position operator is

$$\hat{\mathbf{r}}_{ij}=i\hbar\left(\delta_{ij}\boldsymbol{\nabla}_{\mathbf{p}}-\alpha\delta_{ij}\frac{\mathbf{p}}{p^2}-\sum_{\lambda=1}^{3}(\nabla_{\mathbf{p}}e_{\mathbf{p}\lambda i})e_{\mathbf{p}\lambda j}\right). \tag{3}$$

The subscripts i and j imply that $\hat{\mathbf{r}}_{ij}$ is a 3×3 (generally $n\times n$) matrix that operates on a vector wave function. The last term of Eq. (3) subtracts off contributions in which a unit vector is replaced by its gradient. The Hermiticity of this operator will now be examined and it will be compared to the Pryce operator, (1).

Hermiticity will be considered first. Since $\Sigma e_{\mathbf{p}\lambda i}e_{\mathbf{p}\lambda j}$ is a unit matrix, its gradient is zero and $\Sigma(\nabla_{\mathbf{p}}e_{\mathbf{p}\lambda i})e_{\mathbf{p}\lambda j}$ must be antisymmetric. Thus multiplication by i gives a Hermitian matrix. The position operator $i\hbar\nabla_{\mathbf{p}}-i\hbar\mathbf{p}/2p^2$, appropriate to covariant normalization, applies to Klein-Gordon particles [12], and the extent of its applicability to the photon has been investigated [13]. For a consistent definition of scalar product, these authors concluded that this operator is Hermitian. If $\alpha=0$, then the second term is zero, and the scalar product considered in [8] is obtained. In all cases, integration by parts of $\langle\psi|\hat{\mathbf{r}}|\phi\rangle$ defined by Eq. (2) proves that $\hat{\mathbf{r}}$ given by Eq. (3) is Hermitian.

The position operator was constructed to ensure that each of its components has eigenvectors of the form

$$\psi_{r'\lambda}(\mathbf{p})\sim p^{\alpha}\mathbf{e}_{\mathbf{p}\lambda}e^{-i\mathbf{p}\cdot\mathbf{r}'/\hbar}, \tag{4}$$

where the kth component of $\hat{\mathbf{r}}$ has eigenvalue r'_k. Since it can be proved that operators with simultaneous eigenvectors commute, it follows that this must be the case for the components of $\hat{\mathbf{r}}$. In addition, it was verified by a direct calculation that the components of $\hat{\mathbf{r}}$ are mutually commuting. Thus $\hat{\mathbf{r}}$ generates a basis of transverse and longitudinal localized photon states.

To make a comparison with the Pryce operator, the last term of Eq. (3) must be evaluated. The gradient of the longitudinal unit vector, $\nabla_{\mathbf{p}}(\mathbf{p}/p)=\Sigma_{\lambda=1,2}\mathbf{e}_{\mathbf{p}\lambda}\mathbf{e}_{\mathbf{p}\lambda}/p$, can be obtained by direct evaluation using Cartesian components. Antisymmetry then dictates that $\nabla_{\mathbf{p}}\mathbf{e}_{\mathbf{p}\lambda}=-\mathbf{e}_{\mathbf{p}3}\mathbf{e}_{\mathbf{p}\lambda}/p$ for $\lambda=1$ and 2. But matrix elements that connect $\mathbf{e}_{\mathbf{p}1}$ with $\mathbf{e}_{\mathbf{p}2}$ are also allowed.

To evaluate these transverse to transverse matrix elements, a fixed but arbitrary z axis pointing in the direction $\mathbf{e}_z$ can be defined. The transverse directions can be chosen as the spherical polar θ and ϕ unit vectors, so that $\mathbf{e}_{\mathbf{p}1}\equiv\theta_{\mathbf{p}}$ and $\mathbf{e}_{\mathbf{p}2}\equiv\phi_{\mathbf{p}}$ where the subscript **p** indicates that these vectors are functions of **p**. Only a change in $\phi_{\mathbf{p}}$ couples the two transverse components. This is the $\mathbf{e}_{\mathbf{p}2}$ component of ∇_p. For a rotation about the z axis through $d\phi$, $d\mathbf{e}_{\mathbf{p}1}=\cos\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}d\phi$ and $d\mathbf{e}_{\mathbf{p}3}=\sin\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}d\phi$, so that $\nabla_{\mathbf{p}}\mathbf{e}_{\mathbf{p}1}=\cot\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}\mathbf{e}_{\mathbf{p}2}/p$ and, by antisymmetry, $\nabla_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}=-\cot\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}1}\mathbf{e}_{\mathbf{p}2}/p$. The position operator can thus be written as

$$\begin{aligned}\hat{\mathbf{r}}_{ij}=&\,i\hbar\delta_{ij}\nabla_{\mathbf{p}}-i\hbar\alpha\delta_{ij}\frac{\mathbf{p}}{p^2}-\frac{i\hbar}{p^2}\\&\times\sum_{\lambda=1}^{2}(e_{\mathbf{p}\lambda i}\mathbf{e}_{\mathbf{p}\lambda}p_j-p_i\mathbf{e}_{\mathbf{p}\lambda}e_{\mathbf{p}\lambda j})\\&-\frac{i\hbar}{p}\cot\theta_{\mathbf{p}}(e_{\mathbf{p}2i}\mathbf{e}_{\mathbf{p}2}e_{\mathbf{p}1j}-e_{\mathbf{p}1i}\mathbf{e}_{\mathbf{p}2}e_{\mathbf{p}2j}).\end{aligned} \tag{5}$$

The last two terms of Eq. (5) can be expressed in terms of the **S** matrices. This is most easily done in transverse and longitudinal coordinates. A unitary transformation can then be performed to obtain Cartesian components. The standard spin matrices are $(S_i)_{jk}=-i\epsilon_{ijk}$ where ϵ is the antisymmetric Levi-Civita symbol. The matrix $S_{\mathbf{p}i}=\mathcal{U}^{-1}S_i\mathcal{U}$ rotates a **p**-space vector to Cartesian axes, operates on it with S_i, and rotates it back where

$$\mathcal{U}_{\lambda j}\equiv\mathbf{e}_{\mathbf{p}\lambda}\cdot\mathbf{e}_j. \tag{6}$$

This gives

$$S_{\mathbf{p}1}=\begin{pmatrix}0 & i\sin\theta_{\mathbf{p}} & i\cos\theta_{\mathbf{p}}\sin\phi_{\mathbf{p}}\\ -i\sin\theta_{\mathbf{p}} & 0 & -i\cos\theta_{\mathbf{p}}\cos\phi_{\mathbf{p}}\\ -i\cos\theta_{\mathbf{p}}\sin\phi_{\mathbf{p}} & i\cos\theta_{\mathbf{p}}\cos\phi_{\mathbf{p}} & 0\end{pmatrix}, \tag{7}$$

$$S_{\mathbf{p}2}=\begin{pmatrix}0 & 0 & i\cos\phi_{\mathbf{p}}\\ 0 & 0 & i\sin\phi_{\mathbf{p}}\\ -i\cos\phi_{\mathbf{p}} & -i\sin\phi_{\mathbf{p}} & 0\end{pmatrix}, \tag{8}$$

and

$$S_{\mathbf{p}3}=\begin{pmatrix} 0 & -i\cos\theta_{\mathbf{p}} & i\sin\theta_{\mathbf{p}}\sin\phi_{\mathbf{p}} \\ i\cos\theta_{\mathbf{p}} & 0 & -i\sin\theta_{\mathbf{p}}\cos\phi_{\mathbf{p}} \\ -i\sin\theta_{\mathbf{p}}\sin\phi_{\mathbf{p}} & i\sin\theta_{\mathbf{p}}\cos\phi_{\mathbf{p}} & 0 \end{pmatrix}. \tag{9}$$

The vector spin operator $\mathbf{S}=\Sigma_i \mathbf{e}_{\mathbf{p}i} S_{\mathbf{p}i}$ can also be written as $\mathbf{S}=\Sigma_i \mathbf{e}_i S_i$.

The position operator becomes

$$\hat{\mathbf{r}}=i\hbar\nabla_{\mathbf{p}}-i\hbar\alpha\frac{\mathbf{p}}{p^2}+\frac{\hbar}{p^2}\mathbf{p}\times\mathbf{S}-\frac{\hbar}{p}\cot\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}S_{\mathbf{p}3}. \tag{10}$$

The first three terms of Eq. (10) give precisely the Pryce operator in the case $\alpha=\frac{1}{2}$. However, the components of $\mathbf{r}_P$ do not commute. The operator $\hat{\mathbf{r}}$ found here has an additional term that allows its components to commute, and nature of this term will be discussed in the next section.

III. RELATION TO BERRY'S PHASE

The last term of Eq. (10), $-\hbar\mathbf{a}$, where

$$\mathbf{a}(\mathbf{p})\equiv\frac{1}{p}\cot\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}S_{\mathbf{p}3}, \tag{11}$$

will now be examined. While it was convenient to use spherical polar coordinates in Sec. II, $\mathbf{a}$ can be written in a form that does not depend on the choice of polarization vectors as

$$\mathbf{a}(\mathbf{p})=\frac{1}{p^2}\frac{(\mathbf{p}\cdot\hat{\mathbf{e}}_z)(\hat{\mathbf{e}}_z\times\mathbf{p})(\mathbf{p}\cdot\mathbf{S})}{p^2-(\mathbf{p}\cdot\hat{\mathbf{e}}_z)^2}.$$

This operator contains the contributions to the gradient that couple the two transverse unit vectors in Eq. (3). It is equivalent to the second term of Białynicki-Birula's phase-invariant derivative [14,15],

$$\hat{\mathbf{D}}\equiv\nabla_{\mathbf{p}}+i\lambda\mathbf{a}(\mathbf{p}), \tag{12}$$

where λ is the photon's helicity, $\mathbf{a}\equiv i\Sigma_i e_i^*\nabla_{\mathbf{p}}e_i$, and $e_i\equiv(e_{\mathbf{p}1i}\pm ie_{\mathbf{p}2i})/\sqrt{2}$. They found that $\mathbf{a}$ describes Berry's phase [15], and obtained an expression that agrees with Eq. (11) for spin-1 particles.

The $\mathbf{p}$-space line integral of Eq. (11) over a closed loop about the z axis gives

$$\gamma(C)=\oint\mathbf{a}(\mathbf{p})\cdot d\mathbf{p}=2\pi\cos\theta_{\mathbf{p}} \tag{13}$$

with $d\mathbf{p}=pd\mathbf{e}_{p3}=p\sin\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}d\phi$ as used in the derivation of Eq. (5). This differs from Berry's topological phase [16] only by a term 2π multiplied by an integer, which does not affect $e^{i\gamma(C)}$. More generally $\gamma(\mathbf{p})=\int\mathbf{a}(\mathbf{p}')\cdot d\mathbf{p}'$ is a noncyclic adiabatic phase [17].

Existence of a Berry's phase of the form (13) for photons [18] has been verified experimentally [19]. The physics here is similar: As a localized photon moves about in $\mathbf{p}$ space, its direction of travel changes adiabatically, and it acquires a phase γ. The term with momentum $\mathbf{p}$ thus goes as $e^{i\gamma(\mathbf{p})}$. Mathematically, this looks like a change in its position, but it does not reflect an actual movement of the photon. To keep the contributions of all $\mathbf{p}$'s in phase, this must be eliminated, and this is the function of the last term of Eq. (10). It is the inclusion of this adiabatic phase term in addition to the Pryce $\mathbf{p}\times\mathbf{S}/p^2$ term that distinguishes Eq. (10) from all position operators in the existing literature of which I am aware.

IV. COMMUTATION RELATIONS AND THE POINCARÉ GROUP

The commutation relations will be considered next. With $\hat{\mathbf{r}}$ expressed in terms of the S matrices, the subscript ij can be dropped. In what follows, i and j refer instead to components of the vector operators. The components of $\hat{\mathbf{r}}$ commute,

$$[\hat{r}_i,\hat{r}_j]=0. \tag{14}$$

In $\mathbf{p}$ space, the momentum operator $\hat{\mathbf{p}}=\mathbf{p}$ commutes with all terms of Eq. (10) except the first. Thus the commutation relation satisfied by the ith component of the position operator and the jth component of the momentum operator is

$$[\hat{r}_i,\hat{p}_j]=i\hbar\delta_{ij}. \tag{15}$$

For $\hat{H}\equiv cp$ [8] or $\hat{H}\equiv c\mathbf{p}\cdot\hat{\mathbf{S}}=cp\hat{S}_{\mathbf{p}3}$ [10],

$$[\hat{r}_i,\hat{H}]=i\hbar\hat{H}\frac{p_i}{p^2}. \tag{16}$$

If $\hat{H}=cp$, $d\hat{O}/dt=[\hat{O},\hat{H}]/i\hbar$ implies photon velocity components of $d\hat{r}_i/dt=cp_i/p$, while for $\hat{H}\equiv c\mathbf{p}\cdot\hat{\mathbf{S}}$ they are $c\lambda p_i/p$. All commutation relations in the realization of the Poincaré algebra defined below follow from Eqs. (14)–(16) once angular momentum and boost operators have been defined.

The position operator (10) will be used to define the required angular momentum and boost operators. Operation with $\hat{\mathbf{r}}\times\hat{\mathbf{p}}$ on a state describing a particle localized at $\mathbf{r}'$ gives $\mathbf{r}'\times\mathbf{p}$. Thus it describes only orbital angular momentum, and the nomenclature

$$\hat{\mathbf{L}}\equiv\hat{\mathbf{r}}\times\hat{\mathbf{p}} \tag{17}$$

is appropriate. Boosts are generated by

$$\hat{\mathbf{K}}\equiv\frac{1}{2c}(\hat{\mathbf{r}}\hat{H}+\hat{H}\hat{\mathbf{r}}). \tag{18}$$

Using Eqs. (14)–(18) it can be verified that $\hat{\mathbf{p}}$, $\hat{H}$, $\hat{\mathbf{L}}$, and $\hat{\mathbf{K}}$ satisfy standard commutation relations [20], and thus provide a realization of the Lie algebra of the Poincaré group: $[\hat{L}_i,\hat{L}_j]=i\hbar\epsilon_{ijk}\hat{L}_k$, $[\hat{L}_i,\hat{K}_j]=i\hbar\epsilon_{ijk}\hat{K}_k$, $[\hat{K}_i,\hat{K}_j]=-i\hbar\epsilon_{ijk}\hat{L}_k$, $[\hat{L}_i,\hat{p}_j]=i\hbar\epsilon_{ijk}\hat{p}_k$, $[\hat{K}_i,\hat{p}_j]=i\hbar\delta_{ij}\hat{H}/c$, $[\hat{K}_i,\hat{H}]=i\hbar c\hat{p}_i$, and $[\hat{L}_i,\hat{H}]=[\hat{p}_i,\hat{H}]=0$.

All of these operators are diagonal if expressed in a transverse-longitudinal basis, and thus do not mix transverse and longitudinal components. Only the internal angular momentum operator $\hat{S}_{\mathbf{p}3}$ rotates one transverse component into the other. Its eigenvectors are states of definite helicity. For transverse photon states, the other two spin operators, which would rotate transverse vectors into longitudinal ones, are not part of the algebra. If the Białynicki-Birula [10] Hamiltonian is used, the operators can contain only transverse components. The resulting expression is rather complicated in a Cartesian basis.

Commutation relations involving $\hat{\mathbf{r}}$ and $\hat{S}_{\mathbf{p}3}$ directly are not part of the Poincaré algebra, and they will be derived next. Substitution of Eqs. (14) and (15) gives

$$[\hat{L}_i,\hat{r}_j]=i\hbar\epsilon_{ijk}\hat{r}_k.$$

Using Eqs. (7) to (9), $[\hat{S}_{\mathbf{p}i},\hat{S}_{\mathbf{p}j}]=i\epsilon_{ijk}\hat{S}_{\mathbf{p}k}$, and the gradient of the unit vectors $\mathbf{e}_{\mathbf{p}i}$ used to obtain Eq. (5), it can be proved that

$$[\hat{r}_i,\hat{S}_{\mathbf{p}j}]=0.$$

This implies that simultaneous $\hat{\mathbf{r}}$ and $\hat{S}_{\mathbf{p}3}$ eigenvectors exist that describe a localized photon with definite helicity. Also,

$$[\hat{L}_i,\hat{S}_{\mathbf{p}j}]=0,$$

so that orbital and internal angular momentum are compatible observables.

Using Eqs. (17) and (10), the orbital angular momentum becomes

$$\hat{\mathbf{L}}=-i\hbar\mathbf{p}\times\nabla_{\mathbf{p}}+\hbar\mathbf{S}_{\perp}-\hbar\cot\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}1}S_{\mathbf{p}3}, \tag{19}$$

where $\mathbf{S}_{\perp}=\mathbf{e}_{\mathbf{p}1}S_{\mathbf{p}1}+\mathbf{e}_{\mathbf{p}2}S_{\mathbf{p}2}$. Expression (19) can alternatively be written as

$$\hat{\mathbf{L}}=-i\hbar\mathbf{p}\times\nabla_{\mathbf{p}}+\hbar\mathbf{e}_{\mathbf{p}2}S_{\mathbf{p}2}-\frac{\hbar}{\sin\theta_{\mathbf{p}}}\mathbf{e}_{\mathbf{p}1}S_3,$$

in which S_3 describes a rotation about a fixed, though arbitrary, z axis. The $\sin\theta_{\mathbf{p}}$ dependence comes from $d\mathbf{p}=p\sin\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}d\phi$ for rotation about the z axis. The position operator (10), with transverse components equal to $\mathbf{p}\times\hat{\mathbf{L}}/p^2$, can be written in a similar form.

Inspection of Eqs. (10) and (19) shows that, under the parity transformation $\mathbf{p}\rightarrow-\mathbf{p}$, the position and angular momentum operators transform as $\hat{\mathbf{r}}\rightarrow-\hat{\mathbf{r}}$ and $\hat{\mathbf{L}}\rightarrow\hat{\mathbf{L}}$. Time inversion for which $\mathbf{p}\rightarrow-\mathbf{p}$ and a c number $c\rightarrow c^*$ gives $\hat{\mathbf{S}}\rightarrow-\hat{\mathbf{S}}$, which results in $\hat{\mathbf{r}}\rightarrow\hat{\mathbf{r}}$ and $\hat{\mathbf{L}}\rightarrow-\hat{\mathbf{L}}$. Transformation properties follow from the commutation relations and thus are of the usual form. The unitary transformation $U_{\delta}\equiv e^{-i\mathbf{p}\cdot\delta/\hbar}$ generates a translation $U_{-\delta}\hat{\mathbf{r}}U_{\delta}=\hat{\mathbf{r}}+\delta$ and $U_{\phi\mathbf{e}_z}\equiv e^{-i\phi\hat{\mathbf{L}}\cdot\mathbf{e}_z/\hbar}$ gives $U_{-\phi\mathbf{e}_z}\hat{r}_iU_{\phi\mathbf{e}_z}=\hat{r}_i+\phi(\mathbf{e}_z\times\hat{\mathbf{r}})_i$, which rotates the position operator about the z axis.

The definition (18) can be used to show that

$$\hat{\mathbf{r}}=\frac{c}{2}(\hat{H}^{-1}\hat{\mathbf{K}}+\hat{\mathbf{K}}\hat{H}^{-1}),$$

so that the transformation properties of $\hat{\mathbf{r}}$ are also determined by the Poincaré algebra. Thus the Lorentz transformation properties of $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ derived in [21] are valid here. The time coordinate is not transformed, since it is not an operator in quantum mechanics [21,2]. Thus the transformed position and time describe a different point on the particle's world line.

Starting from the assumption that the components of $\hat{\mathbf{r}}$ commute with each other and with $\hat{S}_{\mathbf{p}i}$, it can be verified that the components of $\hat{\mathbf{p}}_P=\hat{\mathbf{r}}+\hbar\cot\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}S_{\mathbf{p}3}/p$ obey the commutation relation $[\hat{r}_i,\hat{r}_j]=-i\hbar^2\epsilon_{ijk}e_{\mathbf{p}3k}S_{\mathbf{p}3}/p^2$. This is consistent with the findings of Mourad [8], and the fact that $\hat{\mathbf{r}}_P$ does not have commuting components, while $\hat{\mathbf{r}}$ does.

The new position operator $\hat{\mathbf{r}}$ derived here satisfies the conditions placed on a position operator by previous authors [2,3,6,8]. Additionally, it has commuting components as required if its eigenvectors are to describe localized states.

V. DISCUSSION

The new position operator (10) will first be discussed in the context of other position operators in the literature and the widespread claim that a photon position operator with commuting components does not exist. The choice of scalar product and the consequent normalization of the wave functions is somewhat of a side issue. It will be ignored in what follows, and the $\mathbf{p}/p^2$ term will be omitted.

Pryce [2] derived operators for position coordinates weighted by energy ($\hat{\mathbf{q}}$), by mass ($\hat{\mathbf{X}}$), and the linear combination $\hat{\hat{\mathbf{q}}}=(E\hat{\mathbf{q}}+mc^2\hat{\mathbf{X}})/(E+mc^2)$. Of these, only $\hat{\hat{\mathbf{q}}}$ has commuting components. For massive spin-1 particles he obtained $\hat{\hat{\mathbf{q}}}=i\hbar\nabla_{\mathbf{p}}$. For the zero mass photon, he was unable to write an expression for $\hat{\mathbf{X}}$, and was left with $\hat{\mathbf{q}}=\hat{\mathbf{r}}_P$ given by Eq. (1).

Newton and Wigner [3] again obtained $\hat{\hat{\mathbf{q}}}=i\hbar\nabla_{\mathbf{p}}$ for $m>0$, which is unique to within terms required for normalization [22]. They briefly refer to zero mass and state that "no localized states in the above sense exist" in the case of photons. Their paper has been widely quoted as proof that there is no position operator for the photon (with commuting components) [12,24,8,9,25,7,23]. However, the present counterexample implies that this is not the case.

There are two position operators with noncommuting components that have been applied to photons: the Pryce operator $\hat{\mathbf{r}}_P$ and the phase-invariant derivative $i\hbar\hat{\mathbf{D}}$ given by Eq. (12). Their eigenvectors can be localized to arbitrary precision [8]. They provide, respectively, realizations of the Foldy and standard helicity or Moses representations of the Poincaré algebra [15], and are related by a unitary transformation [26].

The Newton-Wigner-Pryce operator $\hat{\mathbf{q}}=i\hbar\nabla_{\mathbf{p}}$ has eigenvectors with fixed directions in space, while $\hat{\mathbf{r}}$ has transverse and longitudinal eigenvectors in $\mathbf{p}$ space. While fixed directions can be used to describe massive particles where all three spin directions exist, this is not possible for the photon, since its two transverse polarization directions do not span three-dimensional space. It has been argued [12,24] that, roughly speaking, a photon is not localizable because only momenta perpendicular to its polarization direction are available. However, all momenta can contribute to the eigenvectors of $\hat{\mathbf{r}}$, Eq. (4), because the basis consists of transverse and longitudinal states, which rotate with the photon's momentum vector.

A position space probability amplitude $\langle \mathbf{r}\lambda|\psi\rangle$ can be calculated using Eqs. (2) and (4). The configuration space wave function is then

$$\psi_{\lambda}(\mathbf{r})\sim\int d^3p\langle \mathbf{p}\lambda|\psi\rangle \mathbf{e}_{\mathbf{p}\lambda}p^{\alpha}e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar},$$

consistent with the claim in [25] that the $\mathbf{r}$-space photon wave function exists as a Fourier transform, since superpositions of solutions must be allowed. Photon wave functions of this form have been used in bounded and inhomogeneous media [10] and the calculation of spontaneous emission by an excited atom [9]. However, the existence of localized eigenvectors of $\hat{\mathbf{r}}$ suggests that, in general, it is not necessary to replace probability density with energy density as proposed in [10] and [9].

Sets of operators satisfying the same commutation relations are unitarily equivalent. Both $\hat{\mathbf{q}}$ and $\hat{\mathbf{r}}$ have commuting components and, together with $\hat{H}$ and $\mathbf{p}$, provide realizations of the same algebra. The eigenvectors of $\hat{\mathbf{q}}$ and $\hat{\mathbf{r}}$, here to be called $\mathbf{f}$ and $\mathbf{g}$, respectively, satisfy $\mathbf{g}=U\mathbf{f}$ where the unitary transformation U rotates transverse and longitudinal vectors to Cartesian axes. Since $U_{\lambda j}=e_{\mathbf{p}\lambda j}$ from Eq. (6),

$$\hat{\mathbf{r}}_{ij}=\sum_{\lambda} U_{i\lambda}^{-1}i\hbar\nabla_{\mathbf{p}}U_{\lambda j}=i\hbar\delta_{ij}\nabla_{\mathbf{p}}+\sum_{\lambda=1}^{3} e_{p\lambda i}(i\hbar\nabla_{\mathbf{p}}e_{\mathbf{p}\lambda j}),$$

which can be seen to be in agreement with Eq. (3) after allowing for antisymmetry. The internal angular momentum $\hbar\mathbf{S}$ used in the description of massive particles [2,3,22] transforms as $U^{-1}S_iU=S_{\mathbf{p}i}$ Eqs. (7)–(9). Thus $\{\hat{\mathbf{r}},S_{\mathbf{p}i},\mathbf{p},\hat{H}\}$ and $\{\hat{\mathbf{q}},\hat{S}_i,\mathbf{p},\hat{H}\}$ and operators derived from them are related by a unitary transformation, and the same commutation relations are obeyed by both of these sets of operators.

Previous authors have looked for position operators that are simple combinations of the momentum and spin operators and have the necessary symmetries [2,3,27,8]. Here the process was reversed, and a position operator with transverse and longitudinal eigenvectors and commuting components was sought. An additional term was found that does not have a simple form, but that has appeared before [26,14,15]. This last term of Eq. (10) compensates for adiabatic phase changes, and is essential to the description of localized photon states.

In summary, a Hermitian photon position operator with commuting components was found. Its eigenfunctions were assumed to be transverse vectors, and the position operator was constructed systematically using this requirement. The position eigenvectors form the basis for a representation in coordinate space. The position operator (10) was found to provide a realization of the Poincaré group. The operators in this group extract position information from the factor $e^{-i\mathbf{p}\cdot\mathbf{r}'}$, but do not change the direction of a transverse state vector. Only the internal spin operator rotates one transverse state into the other. The new photon position operator obtained here is $\hat{\mathbf{r}}=\hat{\mathbf{r}}_P-\hbar\mathbf{a}$, where $\hat{\mathbf{r}}_P$ is the Pryce operator and $\gamma=\int\mathbf{a}\cdot d\mathbf{p}$ is the adiabatic phase.

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